# 16 Implicit Differentiation

## Calculator Free

1. [14 marks: 5, 3, 6]

(a) Given $\left(1+xy^2\right)^2 + \cos(x+y) = 0$, find $dy/dx$.

(b) Given $f(x) = 3^{\sqrt{x}}$, find $f'(x)$.

(c) Consider the curve with equation $2y^3 - 3y^2 - 3x^2 - 12x = 9$. Find the equation of the tangent to the curve that is parallel to the $y$-axis.

## Calculator Free

2. [7 marks: 4, 3] [TISC]

(a) Given that $y = \sqrt{\dfrac{1+2x}{1+x^2}}$, use *logarithmic differentiation* to find an expression for $\dfrac{dy}{dx}$ in terms of $x$ and $y$.

(b) Find the equation of the line which is *perpendicular to the tangent* to this curve at the point (2, 1).

3. [8 marks: 5, 3]

(a) Given that $y = \sqrt{\dfrac{1+\cos(x)}{1-\sin(x)}}$, use *logarithmic differentiation* to find an expression for $\dfrac{dy}{dx}$ in terms of $x$ and $y$.

# Calculator Free

3. (b) Find the equation of the tangent to this curve at the point where $x = 0$.

---

4. [10 marks: 3, 3, 4] [TISC]

A curve has equation given by $x^2 + xy = y^2 - 5$.

(a) Find $\frac{dy}{dx}$ in terms of $x$ and $y$.

(b) The tangent to the curve at the point P $(p, q)$ is parallel to the line $y = x + 5$.
Show that $q = 3p$.

(c) Hence, or otherwise, find the coordinates of the point(s) on the curve with gradient 1.

# Calculator Free

5. [9 marks: 4, 1, 4] [TISC]

A curve has equation $-2x + 4y - y^3 - x^2y = 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2(1+xy)}{(4-x^2-3y^2)}$.

(b) Find (if possible) the gradient of the tangent to the curve at the point (1, 1).

(c) Find the equation of the tangent(s) to the curve at the point where $y = 1$.

# Calculator Free

6. [8 marks: 3, 2, 3] [TISC]

A curve has equation $\sqrt{x+y} = x$.

(a) Find an expression for $\frac{dy}{dx}$

(b) Show that the tangent to this curve at the point (2, 2) is $3x - y = 4$.

(c) Find the point(s) $(a, b)$ on the curve, where $a$ and $b$ are integers, such that the gradient of the curve is 1. Justify your answer.

# Calculator Free

7. [8 marks: 4, 4] [TISC]

A curve has equation $\sin(xy) = -\cos(x)$ for $0 \le x \le \frac{\pi}{2}$ and $-\frac{\pi}{2} \le y \le 0$.

(a) Find an expression for $\frac{dy}{dx}$.

(b) Show that the tangent to this curve at the point where $y = 0$ has equation $y = \frac{2}{\pi}x - 1$.

---

8. [7 marks: 4, 3] [TISC]

(a) Consider the curve with equation $\ln(y + 1) = x\,y$, show that $\frac{dy}{dx} = \frac{y(y+1)}{1-x(y+1)}$.

# Calculator Free

8. (b) Find the equation of the line passing through the point (1, 1) and parallel to the tangent to this curve at the point ($\ln 2$, 1).

---

9. [8 marks: 5, 3] [TISC]

A curve has equation curve $x^2y + \sqrt{3+y^2} = 3$.

(a) Find the equation of the tangent to this curve at the point (–1, 1).

(b) Use the method of incremental change to find the change in $y$ when $x$ changes from –1.00 to –1.01.

# 16 Implicit Differentiation

## Calculator Free

1. [14 marks: 5, 3, 6]

(a) Given $(1+xy^2)^2 + \cos(x+y) = 0$, find $dy/dx$.

$$2(1+xy^2)\times(y^2+2xy\frac{dy}{dx}) - (1+\frac{dy}{dx})\sin(x+y) = 0 \quad ✓✓✓$$
$$2y^2(1+xy^2) + 4xy(1+xy^2)\frac{dy}{dx} - \sin(x+y) - \frac{dy}{dx}\sin(x+y) = 0$$
$$\frac{dy}{dx} = \frac{\sin(x+y) - 2y^2(1+xy^2)}{4xy(1+xy^2) - \sin(x+y)} \quad ✓✓$$

(b) Given $f(x) = 3^{\sqrt{x}}$, find $f'(x)$.

$$f(x) = e^{\ln 3^{\sqrt{x}}} = e^{\sqrt{x}\ln 3} \quad ✓$$
$$f'(x) = \frac{\ln 3}{2\sqrt{x}} e^{\sqrt{x}\ln 3} \quad ✓$$
$$= \frac{3^{\sqrt{x}}\ln 3}{2\sqrt{x}} \quad ✓$$

(c) Consider the curve with equation $2y^3 - 3y^2 - 3x^2 - 12x = 9$. Find the equation of the tangent to the curve that is parallel to the $y$-axis.

$$6y^2\frac{dy}{dx} - 6y\frac{dy}{dx} - 6x - 12 = 0 \quad ✓$$
$$\frac{dy}{dx} = \frac{6x+12}{6y^2-6y} \quad ✓$$

When tangent is parallel to the $y$-axis, $\frac{dy}{dx} \to \infty$.

Hence, $6y^2 - 6y = 0$ ✓

$y = 0$ or $1$ ✓

When $y = 0$, $-3x^2 - 12x = 9 \Rightarrow x = -1, -3$. ✓

$y = 1$, $-3x^2 - 12x = 10 \Rightarrow x = -2 \pm \frac{\sqrt{6}}{3}$. ✓

Hence, tangents have equations: $x = -1$, $x = -3$, $x = -2 \pm \frac{\sqrt{6}}{3}$





## Calculator Free

2. [7 marks: 4, 3] [TISC]

(a) Given that $y = \sqrt{\frac{1+2x}{1+x^2}}$, use logarithmic differentiation to find an expression for $\frac{dy}{dx}$ in terms of $x$ and $y$.

$$\ln y = \frac{1}{2}[\ln(1+2x) - \ln(1+x^2)] \quad ✓$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{2}{1+2x} - \frac{2x}{1+x^2}\right] \quad ✓✓$$
$$\frac{dy}{dx} = y\left[\frac{1}{1+2x} - \frac{x}{1+x^2}\right] \quad ✓$$

(b) Find the equation of the line which is perpendicular to the tangent to this curve at the point (2, 1).

At (2, 1), gradient of tangent $\frac{dy}{dx} = -\frac{1}{5}$ ✓

Hence, perpendicular line has gradient $m = 5$ ✓

$\Rightarrow$ Equation of perpendicular line is $y = 5x - 9$. ✓

---

3. [8 marks: 5, 3]

(a) Given that $y = \sqrt{\frac{1+\cos(x)}{1-\sin(x)}}$, use logarithmic differentiation to find an expression for $\frac{dy}{dx}$ in terms of $x$ and $y$.

$$\ln y = \frac{1}{2}[\ln(1+\cos x) - \ln(1-\sin x)] \quad ✓✓$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{-\sin x}{1+\cos x} + \frac{\cos x}{1-\sin x}\right] \quad ✓✓$$
$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{-\sin x}{1+\cos x} + \frac{\cos x}{1-\sin x}\right] \quad ✓$$

## Calculator Free

3. (b) Find the equation of the tangent to this curve at the point where $x = 0$.

> When $x = 0$, $y = \sqrt{2}$ ✓
> Gradient of tangent $\frac{dy}{dx} = \frac{\sqrt{2}}{2}$ ✓
> $\Rightarrow$ Equation of tangent is $y = \frac{\sqrt{2}}{2}x + \sqrt{2}$. ✓

---

4. [10 marks: 3, 3, 4] [TISC]

A curve has equation given by $x^2 + xy = y^2 - 5$.

(a) Find $\frac{dy}{dx}$ in terms of $x$ and $y$.

> $2x + (y + x\frac{dy}{dx}) = 2y\frac{dy}{dx}$ ✓✓
> $\frac{dy}{dx} = \frac{y+2x}{2y-x}$ ✓

(b) The tangent to the curve at the point P $(p, q)$ is parallel to the line $y = x + 5$. Show that $q = 3p$.

> Since tangent is parallel to $y = x + 5$, $\frac{dy}{dx} = 1$. ✓
> Hence, $2x + y = 2y - x \Rightarrow y = 3x$. ✓
> Therefore, when $x = p$, $y = 3p \Rightarrow q = 3p$. ✓

(c) Hence, or otherwise, find the coordinates of the point(s) on the curve with gradient 1.

> Since $y = 3x$: $x^2 + 3x^2 = 9x^2 - 5$ ✓
> $x = \pm 1$. ✓
> Hence, points are (1, 3) and (−1, −3). ✓✓





## Calculator Free

5. [9 marks: 4, 1, 4] [TISC]

A curve has equation $-2x + 4y - y^3 - x^2y = 0$.

(a) Show that $\frac{dy}{dx} = \frac{2(1+xy)}{(4-x^2-3y^2)}$.

> $-2 + 4\frac{dy}{dx} - 3y^2\frac{dy}{dx} - (x^2\frac{dy}{dx} + 2xy) = 0$ ✓✓
> $\frac{dy}{dx} = \frac{2xy+2}{4-x^2-3y^2}$ ✓✓
> $= \frac{2(1+xy)}{(4-x^2-3y^2)}$

(b) Find (if possible) the gradient of the tangent to the curve at the point (1, 1).

> When $x = 1$, $y = 1$, $\frac{dy}{dx} \to \infty$. ✓

(c) Find the equation of the tangent(s) to the curve at the point where $y = 1$.

> When $y = 1$: $x^2 + 2x - 3 = 0$
> $x = -3, 1$ ✓
> When $x = 1$, $y = 1$: tangent is parallel to the $y$-axis.
> $\Rightarrow$ equation of tangent is $x = 1$. ✓
> When $x = -3$, $y = 1$: $\frac{dy}{dx} = \frac{1}{2}$. ✓
> $\Rightarrow$ equation of tangent is $y - 1 = \frac{1}{2}(x + 3)$
> $y = \frac{1}{2}(x + 5)$ ✓

## Calculator Free

6. [8 marks: 3, 2, 3] [TISC]

A curve has equation $\sqrt{x+y} = x$.

(a) Find an expression for $\frac{dy}{dx}$

$$\frac{1}{2}(x+y)^{-\frac{1}{2}}\left(1 + \frac{dy}{dx}\right) = 1 \quad ✓✓$$
$$\left(1 + \frac{dy}{dx}\right) = 2(x+y)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 2(x+y)^{\frac{1}{2}} - 1 \quad ✓$$

(b) Show that the tangent to this curve at the point (2, 2) is $3x - y = 4$.

When $x = 2, y = 2$: $\frac{dy}{dx} = 3$. ✓

$\Rightarrow$ equation of tangent is $y - 2 = 3(x - 2)$

$y = 3x - 4$ ✓

(c) Find the point(s) $(a, b)$ on the curve, where $a$ and $b$ are <u>integers</u>, such that the gradient of the curve is 1. Justify your answer.

$$\frac{dy}{dx} = 2(x+y)^{\frac{1}{2}} - 1 = 1$$
$$\Rightarrow \quad (x+y)^{\frac{1}{2}} = 1 \quad ✓$$

Since, $x$ and $y$ must be integers, possible answers are (0, 1) or (1, 0).

But (0, 1) is not on the curve. ✓

(1, 0) is on the curve. $\Rightarrow$ $a = 1, b = 0$ ✓





## Calculator Free

7. [8 marks: 4, 4] [TISC]

A curve has equation $\sin(xy) = -\cos(x)$ for $0 \le x \le \frac{\pi}{2}$ and $-\frac{\pi}{2} \le y \le 0$.

(a) Find an expression for $\frac{dy}{dx}$.

$$\left(x\frac{dy}{dx} + y\right)\cos(xy) = \sin(x) \quad ✓✓✓$$
$$\frac{dy}{dx} = \frac{1}{x}\left(\frac{\sin(x)}{\cos(xy)} - y\right) \quad ✓$$

(b) Show that the tangent to this curve at the point where $y = 0$ has equation $y = \frac{2}{\pi}x - 1$.

$y = 0 \Rightarrow \sin 0 = -\cos x$

$\cos x = 0$

$x = \frac{\pi}{2}$ ✓✓

At $(\frac{\pi}{2}, 0)$: $\frac{dy}{dx} = \frac{2}{\pi}$. ✓

Hence, equation of tangent is $y = \frac{2}{\pi}x - 1$. ✓

8. [7 marks: 4, 3] [TISC]

(a) Consider the curve with equation $\ln(y + 1) = xy$, show that $\frac{dy}{dx} = \frac{y(y+1)}{1-x(y+1)}$.

$$\frac{1}{y+1}\frac{dy}{dx} = x\frac{dy}{dx} + y \quad ✓✓$$
$$\frac{dy}{dx}\left[\frac{1}{y+1} - x\right] = y \quad ✓$$
$$\frac{dy}{dx}\left[\frac{1-x(y+1)}{y+1}\right] = y \quad ✓$$
$$\frac{dy}{dx} = \frac{y(y+1)}{1-x(y+1)}$$

## Calculator Free

8. (b) Find the equation of the line passing through the point (1, 1) and parallel to the tangent to this curve at the point ($\ln 2$, 1).

> Gradient $m = \dfrac{2}{1-2\ln 2}$ ✓
>
> Hence, equation of tangent is
>
> $y - 1 = \dfrac{2}{1-2\ln 2}(x-1)$ ✓
>
> $y = \dfrac{2x}{1-2\ln 2} - \left(\dfrac{1+2\ln 2}{1-2\ln 2}\right)$ ✓

---

9. [8 marks: 5, 3] [TISC]

A curve has equation curve $x^2y + \sqrt{3+y^2} = 3$.

(a) Find the equation of the tangent to this curve at the point (−1, 1).

> ✓ ✓ ✓
>
> $2xy + x^2\dfrac{dy}{dx} + \dfrac{1}{2}(3+y^2)^{-\frac{1}{2}}\left(2y\dfrac{dy}{dx}\right) = 0$
>
> Subst. $x = -1, y = 1$
>
> $-2 + \dfrac{dy}{dx} + \dfrac{1}{2}\dfrac{dy}{dx} = 0$
>
> $\dfrac{dy}{dx} = \dfrac{4}{3}$ ✓
>
> Equation of tangent is:
>
> $y - 1 = \dfrac{4}{3}(x+1)$
>
> $y = \dfrac{4x}{3} + \dfrac{7}{3}$ ✓

(b) Use the method of incremental change to find the change in $y$ when $x$ changes from −1.00 to −1.01.

> $\delta y \approx \left.\dfrac{dy}{dx}\right|_{x=-1, y=1} \times \delta x$ ✓
>
> $\approx \dfrac{4}{3} \times (-0.01) \approx -0.013$ ✓✓





# 17 Applications of Differentiation I

## Calculator Free

1. [13 marks: 3, 2, 4, 4] [TISC]

(a) The expression $\dfrac{x^3+1}{x^2-1}$ can be rewritten as $px + \dfrac{qx+r}{x^2-1}$. Find $p$, $q$ and $r$.

> $x^3 + 1 = px(x^2-1) + qx + r$
>
> Compare $x^3$ coefficient: $p = 1$ ✓
>
> Subst. $x = 0$: $r = 1$ ✓
>
> Subst. $x = 1$: $q + r = 2$
>
> $\Rightarrow q = 1$ ✓

(b) State the equations of all the asymptotes of the curve $y = \dfrac{x^3+1}{x^2-1}$

> Vertical asymptote: $x = 1$ ✓
>
> Oblique asymptote: $y = x$ ✓

(c) The curve with equation $y = \dfrac{x^3+1}{x^2-1}$ has a maximum point at (0, −1).

Use Calculus to show that the curve has a local minimum point at (2, 3).

> $\dfrac{dy}{dx} = \dfrac{3x^2(x^2-1)-(x^3+1)(2x)}{(x^2-1)^2}$ ✓
>
> When $x = 2$, $\dfrac{dy}{dx} = 0$. ✓

| $x$ | $2^-$ | 2 | $2^+$ |
|---|---|---|---|
| $dy/dx$ | − | 0 | + |

> ✓
>
> Hence, (2, 3) is a minimum point. ✓